A property of p-groups of nilpotency class $p+1$ related to a theorem of Schur

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Abstract

In a p-group G of nilpotency class at most $p+1$, we prove that the exponent of the commutator subgroup $\gamma_2(G)$ divides the exponent of $G/Z(G)$. As a consequence, we deduce that the exponent of the Schur multiplier divides the exponent of G for a p-group of nilpotency class at most p, odd order nilpotent groups of class at most 5, center-by-metabelian groups of exponent p, and a class of groups which includes p-groups of maximal class and potent p-groups.

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1. Introduction

A well-known theorem of Schur states that, given a group G, if the central quotient $G/Z(G)$ is finite, then the commutator subgroup $\gamma_2(G)$ is finite. In [32, Theorem 5.3], Neumann proved the same result. The study of how the structure of $G/Z(G)$ determines that of $\gamma_2(G)$ has attracted many researchers. In [33, Theorem 1], Podoski and Szegedy proved that in a finite group G, if $G/Z(G)$ has rank r, then $|\gamma_2(G)| \leq |G : Z(G)|^{r+1}$. In [6, Corollary 3.3], Donadze et al. proved that if $G/Z(G)$ belongs to the class of finite, polycyclic, polycyclic-by-finite, supersolvable or finite p-groups, then $\gamma_2(G)$

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belongs to the same class. In [20, Theorem 1], Mann proved that if $G/Z(G)$ is locally finite with exponent n, then $\gamma_2(G)$ is locally finite with finite exponent bounded by a function $f(n)$ depending only on n. A closely related problem is regarding the exponent of the second homology group $H_2(G,\mathbb{Z})$ of G, where the action of G on $\mathbb{Z}$ is trivial. The problem asks whether or not a given finite group G satisfies

$$\exp(H_2(G,\mathbb{Z}))|\exp(G). \tag{1.0.1}$$

A standard argument [4, Theorem 10.3] shows that if the Sylow p-subgroups of G satisfy (1.0.1), then G does. To see the connection of (1.0.1) with the above theorem, consider a covering group H of G. If we prove

$$\exp(\gamma_2(H)) \mid \exp(H/Z(H)), \tag{1.0.2}$$

then (1.0.1) holds for G since $\exp(H/Z(H)) \mid \exp(G)$ and $\exp(H_2(G,\mathbb{Z})) \mid \exp(\gamma_2(H))$. Lubotzky and Mann showed that (1.0.1) holds for powerful p-groups [18], Jones proved that (1.0.1) holds for groups of class 2 [13], Moravec showed that (1.0.1) holds for metabelian p-groups of exponent p, groups of nilpotency class at most 3, groups of nilpotency class 4 and odd order and p-groups of class at most $p-2$, potent p-groups and p-groups of maximal class ([26], [27], [28], [29] and [30]) and some other classes of groups. In [21], Mashayekhy et al. proved (1.0.1) for p-groups of class less than or equal to $p-1$. The general validity of (1.0.1) was disproved by Bayes et al. in [2]. They gave an example of a 2-group G of class 4 such that $\exp(G) = 4$ and $\exp(H_2(G,\mathbb{Z})) = 8$. The main goal of this paper is to prove the following result:

Theorem 2.4. Let p be a prime and G be a p-group. If the nilpotency class of G is less than or equal to $p+1$, then $\exp(\gamma_2(G)) \mid \exp(G/Z(G))$.

Dealing with the Schur multiplier, it became a common strategy to consider the families of p-groups in the growing order of complexity from abelian to powerful, to small-class and potent. The p-groups of class p lie at the boundary of this sequence, and Theorem 2.4 proves (1.0.1) for this class of groups. As an application, we will prove (1.0.1) for other classes of groups. After writing this article, the third author had some private communication about groups of exponent 5 with Vaughan-Lee, who after some time communicated the important discovery of some new counterexamples to (1.0.1) (cf. [38]). Among them, there is a 5-group G of class 9 such that $\exp(G) = 5$ and

$\exp(H_2(G,\mathbb{Z})) = 25$, as well as a 3-group G of class 9 such that $\exp(G) = 9$ and $\exp(H_2(G,\mathbb{Z})) = 27$. We recall that the counterexample of Bayes et al. is a 2-group of class 4, on the other hand, when p is an odd prime, the above discussion proves (1.0.1) for p-groups of class 5 with the exception of the case $p = 3$. Our second goal is to fill this gap. In fact we obtain a bound for the exponent of the exterior square $G \wedge G$. This group, introduced by Miller [23], is isomorphic with the commutator subgroup $\gamma_2(H)$ of any covering group H of G. Moreover, it is an elementary fact of Schur's theory that the following statements are equivalent:

a) If a group G has class at most $c+1$, then $\exp(\gamma_2(G)) \mid \exp(G/Z(G))$.

b) If a group H has class at most c, then $\exp(H \wedge H) \mid \exp(H)$.

The case of p-groups of class 5 is settled by the following result:

Theorem 3.5. If G is a finite 3-group of class at most 5, then $\exp(G \wedge G)$ divides $\exp(G)$.

A common line of research has been to give a bound for the exponent of the Schur multiplier depending on the nilpotency class. Towards this we prove the following theorem which improves the bound obtained by Sambonet [36, Theorem 1.1] for an odd prime p, and consequently the bounds given by Moravec [26], and Ellis [10].

Theorem 4.2. If p is an odd prime and G is a p-group of class $c \geq p$, then $\exp(G \wedge G) \mid (\exp(G))^n$, for $n = \lceil \log_{p-1}(\frac{c+1}{p+1}) \rceil + 1$.

If $p = 3$, we further improve the bound given in the above theorem in Lemma 4.6 by proving $\exp(G \wedge G) \mid (\exp(G))^n$, where $n = \lceil \log_3(\frac{c+1}{2}) \rceil$.

The paper is organized as follows. In Section 2 we prove Theorem 2.4, and here the main ingredients are the Hall's commutator collection formulae [16, Proposition 1.1.32], a Lemma of Mann [19], and the identity (2.4.1) which we obtain using multilinearity. In Section 3 we prove Theorem 3.5, to this aim we use the notion of non-abelian tensor product due to Brown and Loday [5], and the following exact sequence of Ellis for the exterior product [10]: given $N, M \trianglelefteq G$ with $M \leq N$, we have

$$M \wedge G \to N \wedge G \to \frac{N}{M} \wedge \frac{G}{M} \to 1, \qquad (1.0.3)$$

and in particular

$$\exp(N \wedge G) \mid \exp(M \wedge G) \cdot \exp[(N/M) \wedge (G/M)]. \tag{1.0.4}$$

In Section 4 we prove Theorem 4.2 and some bounds on the exponent of the Schur multiplier depending on the derived length (cf. Lemma 4.9). The results of Section 4 are also proved using (1.0.4).

2. The Schur multiplier of p-groups having nilpotency class p

Let G be a group. In this paper we use the right notation, and the commutators are left normed. We denote the i-th term of the lower central series of G by $\gamma_i(G)$. A standard argument shows that in a group G, any commutator of weight r is multilinear modulo $\gamma_{r+1}(G)$, in particular

$$[g_1, \ldots, x_i y_i, \ldots, g_r] \equiv [g_1, \ldots, x_i, \ldots, g_r][g_1, \ldots, y_i, \ldots, g_r] \mod \gamma_{r+1}(G)$$

for all $i = 1, \ldots, r$. Now we recall the collection formulae of P. Hall [16, Proposition 1.1.32].

Theorem 2.1 (Commutator collection formulae)**.** *Let x and y be elements of G, and let p be a prime and r a positive integer. For $a, b \in \langle x, y\rangle$ define $K(a, b)$ to be the normal closure in $\langle x, y\rangle$ of the set of all basic commutators in $\{a, b\}$ of weight at least p^r and of weight at least 2 in b, together with the p^{r-k+1}-th powers of all basic commutators in $\{a, b\}$ of weight less than p^k and of weight at least 2 in b for $1 \leq k \leq r$. Then:*

(*i*) $(xy)^{p^r} \equiv x^{p^r} y^{p^r} [y, x]^{\binom{p^r}{2}} [y, {}_2\, x]^{\binom{p^r}{3}} \cdots [y, {}_{p^r-1}\, x] \mod K(x, y)$.

(*ii*) $[x^{p^r}, y] \equiv [x, y]^{p^r} [x, y, x]^{\binom{p^r}{2}} \cdots [x, y, {}_{p^r-1}\, x] \mod K(x, [x, y])$.

Recall that p-groups of class at most $p-1$ are regular, which have a nice power-commutator structure. A p-group of class p need not be regular, but they share a nice property as discovered by Mann [19].

Lemma 2.2. (*Mann*) *Let G be a p-group of class $c \leq p$, and let $x, y \in G$. Then $[x, y^{p^n}] = 1$ is equivalent to $[x, y]^{p^n} = 1$ and, similarly, it is equivalent to $[x^{p^n}, y] = 1$.*

We remark that the original version of this lemma is stated for $n = 1$, nonetheless the case $n > 1$ requires a minor modification, namely the use of [3, Theorem 7.2(a)]. Moreover, Lemma 2.2 and [3, Theorem 7.2(b)] readily prove that any p-group G having class at most p satisfies (1.0.2), that is to say, any p-group G having class at most $p-1$ satisfies (1.0.1), which is a result due to Mashayeky et al. [21]. We remark that this also covers a result of Moravec in [28] that p-groups of class at most $p-2$ satisfy (1.0.1).

Using the collection formulae together with the above lemma we obtain the following result:

Lemma 2.3. *Let x and y be elements of a p-group G such that $\langle x, y\rangle$ has class at most $p+1$. Then $[g, {}_p\, x]^{-1} = [x, g, {}_{p-1}\, x]$ for all g in $\langle x, y\rangle$. Moreover if $[x^{p^n}, y] = 1$, then $[x, g]^{p^n} = [g, {}_p\, x]^{\binom{p^n}{p}}$ for all g in $\langle x, y\rangle$.*

Proof. Clearly we can assume that $G = \langle x, y\rangle$. Since $\gamma_{p+2}(G) = 1$, the commutators of weight $p+1$ are multilinear and central. Therefore, it is enough to consider the case $g = y$. Moreover, $[x, y, {}_{p-1}\, x][y, {}_p\, x] = [xy, xy, {}_{p-1}\, x] = 1$ which proves the first equality. Now we analyze how the assumption $[x^{p^n}, y] = 1$ affects the second commutator collection formula.

Case (i) $p = 2$. Applying Theorem 2.1(*ii*), as $K(x, [x, y]) \leq \gamma_4(G) = 1$, we readily get $[x, y]^{2^n} = [y, x, x]^{\binom{2^n}{2}}$.

Case (ii) $p > 2$. First observe that for $k > p$ the terms $[x, y, {}_{k-1}\, x]^{\binom{p^n}{k}} = 1$, since they lie in $\gamma_{p+2}(G)$. Furthermore the commutators in $\{x, [x, y]\}$ of weight p and of weight at least 2 in $[x, y]$ are in $\gamma_{p+2}(G) = 1$. Therefore $K(x, [x, y])$ becomes the normal closure in $H = \langle x, [x, y]\rangle$ of the set of p^n-th powers of all basic commutators in $\{x, [x, y]\}$ of weight less than p and of weight at least 2 in $[x, y]$. Note that H has class at most p and $x^{p^n} \in Z(H)$, hence every commutator in H having weight at least 1 in x has order dividing p^n by Lemma 2.2. Thus $K(x, [x, y]) = 1$, and also $[x, y, {}_{k-1}\, x]^{\binom{p^n}{k}} = 1$ for $2 \leq k \leq p-1$, as p^n divides $\binom{p^n}{k}$. Therefore Theorem 2.1(*ii*) gives $[x, y]^{p^n}[x, y, {}_{p-1}\, x]^{\binom{p^n}{p}} = 1$, yielding $[x, y]^{p^n} = [y, {}_p\, x]^{\binom{p^n}{p}}$. □

Theorem 2.4. *Let p be a prime and G be a p-group. If the nilpotency class of G is less than or equal to $p+1$, then $\exp(\gamma_2(G)) \mid \exp(G/Z(G))$.*

Proof. Let $\exp(G/Z(G)) = p^n$. Note that $\gamma_2(G)$ has class at most $p-1$. Thus using [3, Theorem 7.2(b)], it is sufficient to prove that $[x, y]^{p^n} = 1$ for all $x, y \in G$. Recall that $[x,y]^{p^n} = [y,\ _p\ x]^{\binom{p^n}{p}}$ for all x, y in G by Lemma 2.3.

Case (i) $p = 2$. Since $[x,y]^{2^n} \in Z(G)$, we obtain $[x,y]^{2^n} = [xy,y]^{2^n}$. Then Lemma 2.3 gives $[y,x,x]^{\binom{2^n}{2}} = [y,xy,xy]^{\binom{2^n}{2}}$. Furthermore, since $\gamma_4(G) = 1$, $[y,xy,xy] = [y,x,x][y,x,y]$ by multilinearity. Therefore we obtain $[y,x,y]^{\binom{2^n}{2}} = 1$. Since $[x,y,y] = [y,x,y]^{-1}$, by Lemma 2.3 we obtain $[y,x]^{2^n} = 1$, and hence $[x,y]^{2^n} = 1$.

Case (ii) $p > 2$. We denote by $S(x,y)$ the set of all commutators of type $[y,x,c_1,\dots,c_{p-1}]$ where $c_i \in \{x,y\}$, notice that these elements commute pairwise as $S(x,y) \subset \gamma_{p+1}(G) \subset Z(G)$. In particular, for $0 \le r \le p-1$, we denote by $e_r(x,y)$ the product of all the commutators in $S(x,y)$ of weight $r+1$ in y, this product is well defined and it does not depend on the ordering of the elements. Moreover, since $\gamma_{p+2}(G) = 1$, the commutators of weight $p+1$ are multilinear, thus every element of $S(xy,y)$ can be written as a product of elements in $S(x,y)$. In more details, given $c = [y,xy,c_1,\dots,c_{p-1}] = [y,x,c_1,\dots,c_{p-1}] \in S(xy,y)$ of weight $t+1$ in y, and $[y,x,d_1,\dots,d_{p-1}] \in S(x,y)$ where $d_{i_1} = \dots = d_{i_{p-1-s}} = x$, $d_i = y$ for $i \notin \{i_1,\dots,i_{p-1-s}\}$, we observe that $[y,x,d_1,\dots,d_{p-1}]$ is a factor in writing c as a product of elements in $S(x,y)$ if and only if $c_{i_1} = \dots = c_{i_{p-1-s}} = xy$. Note that $c_{i_1} = \dots = c_{i_{p-1-s}} = xy$ if and only if $c_i = y$ for t indices i, $i \notin \{i_1,\dots,i_{p-1-s}\}$. Therefore $[y,x,d_1,\dots,d_{p-1}]$ comes $\binom{s}{t}$ times as a factor in writing $e_t(xy,y)$ as a product of elements of $S(x,y)$. Thus for all $0 \le t \le p-1$, we obtain

$$e_t(xy,y) = \prod_{s=t}^{p-1} e_s(x,y)^{\binom{s}{t}}. \tag{2.4.1}$$

Since $e_0(x,y) = [y,\ _p\ x]$, by Lemma 2.3 we have $e_0(x,y)^{\binom{p^n}{p}} = e_0(xy,y)^{\binom{p^n}{p}}$. Thus by (2.4.1)

$$\prod_{r=1}^{p-1} e_r(x,y)^{\binom{p^n}{p}} = 1. \tag{2.4.2}$$

For all positive integers m and h, we define recursively the positive integers

$$\alpha_1(h) = 1,\ \alpha_{m+1}(h) = \sum_{k=m}^{h-1} \binom{h}{k} \alpha_m(k)$$

and we claim that for all $x, y \in G$, $1 \le m \le p-1$

$$\prod_{t=m}^{p-1} e_t(x,y)^{\alpha_m(t)\binom{p^n}{p}} = 1. \tag{2.4.3}$$

To see this, we work by induction on m: the case $m = 1$ has been settled in (2.4.2), and by (2.4.1) we have

$$\begin{aligned}
\prod_{t=k}^{p-1} e_t(xy,y)^{\alpha_k(t)} &= \prod_{t=k}^{p-1}\prod_{s=t}^{p-1} e_s(x,y)^{\binom{s}{t}\alpha_k(t)} \\
&= \Big(\prod_{t=k}^{p-1} e_t(x,y)^{\alpha_k(t)}\Big)\Big(\prod_{t=k}^{p-2}\prod_{s=t+1}^{p-1} e_s(x,y)^{\binom{s}{t}\alpha_k(t)}\Big) \\
&= \Big(\prod_{t=k}^{p-1} e_t(x,y)^{\alpha_k(t)}\Big)\Big(\prod_{s=k+1}^{p-1} e_s(x,y)^{\sum_{t=k}^{s-1}\binom{s}{t}\alpha_k(t)}\Big) \\
&= \Big(\prod_{t=k}^{p-1} e_t(x,y)^{\alpha_k(t)}\Big)\Big(\prod_{s=k+1}^{p-1} e_s(x,y)^{\alpha_{k+1}(s)}\Big).
\end{aligned}$$

By induction hypothesis, $\prod_{t=k}^{p-1} e_t(x,y)^{\alpha_k(t)\binom{p^n}{p}} = 1$ and $\prod_{t=k}^{p-1} e_t(xy,y)^{\alpha_k(t)\binom{p^n}{p}} = 1$, hence we obtain (2.4.3) for $m = k+1$. In particular, by taking $m = p-1$, we have $e_{p-1}(x,y)^{\alpha_{p-1}(p-1)\binom{p^n}{p}} = 1$. On the other hand $\alpha_{p-1}(p-1) = (p-1)!$ is prime to p, hence we have $e_{p-1}(x,y)^{\binom{p^n}{p}} = 1$. Finally, since $e_{p-1}(y,x)^{-1} = [y, {}_p\, x]$, Lemma 2.3 gives $[x,y]^{p^n} = 1$. □

By the equivalence noted in the introduction, Theorem 2.4 can be rephrased in terms of the exterior square, a fact which proves (1.0.1) for p-groups of class at most p. We record this as the next corollary.

Corollary 2.5. *If a p-group G has class at most p, then the exponent of $G \wedge G$ divides the exponent of G, therefore G satisfies* (1.0.1)*.*

In [27] and [26], Moravec proved (1.0.1) for groups of exponent 3 and metabelian p-groups of exponent p, respectively. Recall that groups of exponent 3 are of class at most 3 [17], and hence metabelian. It is worth noting that a metabelian group of exponent p is of class at most p by a well-known result due to Meier-Wunderli [22], hence the above mentioned results of [27] and [26] can be obtained from Corollary 2.5. Moreover, center-by-metabelian groups of exponent p and p-central metabelian p-groups have class at most p (see [24, Theorem 3.8] and [15, Theorem 13]), and hence (1.0.1) holds for these classes of groups.

3. The exterior product and the non-abelian tensor product

Now we define the non-abelian tensor product and exterior square of groups. Let G and H be groups that act on themselves by conjugation and each of which acts on the other. The mutual actions are said to be compatible if $g^{h^{g_1}} = g^{g_1^{-1}hg_1}$ and $h^{g^{h_1}} = h^{h_1^{-1}gh_1}$ for all $g, g_1 \in G$ and $h, h_1 \in H$. The non-abelian tensor product of groups is defined for a pair of groups that act on each other provided the actions satisfy the compatibility conditions, and is the group $G \otimes H$ generated by the symbols $g \otimes h$ for $g \in G$, $h \in H$ with the relations

$$gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h), \tag{3.0.1}$$

$$g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1}), \tag{3.0.2}$$

for all $g, g_1 \in G$ and $h, h_1 \in H$. There exists a homomorphism $\kappa : G \otimes G \to G'$ sending $g \otimes h$ to $[g, h]$. Let $\nabla(G)$ denote the subgroup of $G \otimes G$ generated by the elements $x \otimes x$ for $x \in G$. The exterior square of G is defined as $G \wedge G = (G \otimes G)/\nabla(G)$.

Let $N \trianglelefteq G$ and $[g, n, n] = 1$ for all $g \in G, n \in N$. Following our notations, Lemma 4 of [10] becomes

$$n^t \otimes g = (n \otimes [g, n]^{\binom{t}{2}})(n \otimes g)^t \tag{3.0.3}$$

for all integers $t \geq 2$.

Lemma 3.1. *Suppose N, M are normal subgroups of a group G, where N is abelian. Then* $\exp(N \otimes M)$ *divides* $\exp(N)$ *if* $\exp(N)$ *is odd, and* $2\exp(N)$ *if* $\exp(N)$ *is even.*

Proof. Since N is abelian, $[G, N, N] = 1$. Let $\exp(N) = e$, where e is odd. Expanding $n^e \otimes m$ according to (3.0.3) and noting that $[m, n]^{\binom{e}{2}} = 1$, as e is odd, we obtain $(n \otimes m)^e = 1$ for all $n \in N$ and $m \in M$. Furthermore, since N is abelian, it follows from [7] that the nilpotency class of $N \otimes M$ is at most 2. Hence for all $n_1, n_2 \in N$, $m_1, m_2 \in M$, $((n_1 \otimes m_1)(n_2 \otimes m_2))^e =$

$(n_1 \otimes m_1)^e (n_2 \otimes m_2)^e [n_2 \otimes m_2, n_1 \otimes m_1]^{\binom{e}{2}}$. Recall that $[n_2 \otimes m_2, n_1 \otimes m_1] = [n_2, m_2] \otimes [n_1, m_1]$, so we obtain $\exp(N \otimes M) \mid e$. When e is even the proof is similar. □

A normal subgroup N of a p-group G is said to be powerfully embedded in G if $[N, G] \subset N^p$, when p is odd and $[N, G] \subset N^4$, when $p = 2$, and G is powerful if G is powerfully embedded in itself. In [18], Lubotzky and Mann proved (1.0.1) for powerful p-groups. If a p-group G is regular or powerful, then the subgroup G^p of G is the set of all p-th powers of elements of G and G^p is powerful. Suppose now that N is a powerful and normal subgroup of a group G of exponent p^e, where p is odd, then $N \,/\, N^p$ is abelian and $\exp(N^p) = p^{e-1}$. Using induction on e, by Lemma 3.1 and the bound (1.0.4) for $M = N^p$ we have the following result which extends [25, Theorem 3.11] in two directions relaxing the hypothesis that N is powerfully embedded in G, and proving a bound for $\exp(N \wedge G)$ not only for the exponent of the multiplier of the pair (G, N). More about the multiplier of a pair can be found in [9] and [8].

Proposition 3.2. *Let p be an odd prime and N be a normal subgroup of a finite group G. If N is a powerful p-group, then* $\exp(N \wedge G) \mid \exp(N)$.

Using the bound (1.0.4) for $M = G^p$ and $N = G$, Proposition 3.2 and Corollary 2.5, we obtain the following lemma:

Lemma 3.3. *Let p be an odd prime and G be a finite p-group with* $\exp(G) = p^n$. *Suppose G satisfies the following conditions:*

(*i*) *G^p is powerful.*

(*ii*) $\exp(G^p) = p^{n-1}$.

(*iii*) $\gamma_{p+1}(G) \leq G^p$.

Then $\exp(G \wedge G) \mid \exp(G)$. *In particular, G satisfies* (1.0.1).

For example, the conditions of the above lemma are satisfied if G is a p-group satisfying either $\gamma_{p-1}(G) \leq G^p$ or $\gamma_p(G) \leq G^{p^2}$ (see [1, Theorem 2], [12, Corollary 4.7], and [39, Theorem 3.4]). Moreover, Lemma 3.3 gives an alternative proof of [30, Theorem 1.4]. In details, for a p-group G of maximal class with $|\,G\,| \geq p^{p+2}$, where p is an odd prime, it is worth noting that the fundamental subgroup G_1 of G is regular [3, Theorem 9.6(b), Theorem 9.8(a)] and $\gamma_p(G) = G_1^p = G^p$ [3, Theorem 9.6(a) and Exercise 2, p. 119], hence $\exp(G \wedge G) \mid \exp(G)$. Moreover, if $|\, G\,| \leq p^{p+1}$, then $\exp(G \wedge G) \mid \exp(G)$ by Corollary 2.5. If G is a 2-group of maximal class, then G is metacyclic, so $\exp(G \wedge G) \mid \exp(G)$ [14, Theorem 1.2].

In the rest of this section, we aim to prove Theorem 3.5 mentioned in the introduction. We recall the following expansion in a nilpotent group of class 5 obtained from the table given on page 495 of [31].

$$\begin{aligned}(xy)^n =& x^n y^n [y,x]^{\binom{n}{2}} [y,x,x]^{\binom{n}{3}} [y,x,y]^{\binom{n}{2}+2\binom{n}{3}} [y,x,x,x]^{\binom{n}{4}} \\ & [y,x,x,y]^{2\binom{n}{3}+3\binom{n}{4}} [y,x,y,y]^{2\binom{n}{3}+3\binom{n}{4}} [y,x,x,[y,x]]^{\binom{n}{3}+7\binom{n}{4}+6\binom{n}{5}} \\ & [y,x,y,[y,x]]^{6\binom{n}{3}+18\binom{n}{4}+12\binom{n}{5}} [y,x,x,x,x]^{\binom{n}{5}} \\ & [y,x,x,x,y]^{3\binom{n}{4}+4\binom{n}{5}} [y,x,x,y,y]^{\binom{n}{3}+6\binom{n}{4}+6\binom{n}{5}} [y,x,y,y,y]^{3\binom{n}{4}+4\binom{n}{5}}.\end{aligned} \tag{3.3.1}$$

Suppose $N, M \trianglelefteq G$, then Hall's collection formulae yields

$$[N^p, M] \leq [N, M]^p [M, {}_p N]. \tag{3.3.2}$$

The next technical lemma helps to prove that if G is a 3-group of class 5 with $\exp(G) = 3^n$, then $\exp(G^3) = 3^{n-1}$.

Lemma 3.4. *Let G be a 3-group of class at most 5 and $\exp(G) = 3^n \geq 9$. If $x \in G^3$ has order 3^{n-1}, then $(xy^3)^{3^{n-1}} = 1$ for all $y \in G$.*

Proof. First note that $(xy^3)^{3^{n-1}} \equiv x^{3^{n-1}} y^{3^n} \mod [G^3, G^3]^{3^{n-1}} [G^3, G^3, G^3]^{3^{n-2}}$ by (3.3.1). Now to prove $(xy^3)^{3^{n-1}} = 1$, we show that $[G^3, G^3]^{3^{n-1}} = 1$ and $[G^3, G^3, G^3]^{3^{n-2}} = 1$. By (3.3.2) we have $[G^3, G^3] \leq [G, G^3]^3 [G^3, {}_3\, G]$. Now Hall's collection formulae gives $[G, G^3] \leq [G, G]^3 \gamma_4(G)$, which yields $[G, G^3]^3 \leq ([G, G]^3)^3 \gamma_4(G)^3$ as $[\gamma_2(G), \gamma_4(G)] = 1$. Note that $\gamma_2(G)$ has class 2, hence $\gamma_2(G)$ is regular. Thus $(H^{3^i})^{3^j} = H^{3^{i+j}}$ for every subgroup H of $\gamma_2(G)$ [16, Lemma 1.2.12(ii)]. Hence we obtain $[G, G^3]^3 \leq [G, G]^9 \gamma_4(G)^3$. Moreover, as $\gamma_6(G) = 1$, repeated use of (3.3.2) yields $[G^3, {}_3\, G] \leq \gamma_4(G)^3$. Therefore, we obtain

$$[G^3, G^3] \leq [G, G]^9 \gamma_4(G)^3. \tag{3.4.1}$$

Using (3.4.1) we get $[G^3, G^3, G^3] \leq [[G, G]^9 \gamma_4(G)^3, G^3]$. Since $\gamma_6(G) = 1$, expanding $[[G, G]^9 \gamma_4(G)^3, G^3]$ yields $[G^3, G^3, G^3] \leq [[G, G]^9, G^3][\gamma_4(G)^3, G^3]$.

Furthermore $[\gamma_4(G)^3, G^3] \leq (\gamma_5(G)^3)^3$, and $(\gamma_5(G)^3)^3 = \gamma_5(G)^9$ by [16, Lemma 1.2.12(ii)]. Using (3.3.2), we deduce $[[G,G]^9, G^3] \leq [G,G,G^3]^9$ yielding

$$[G^3, G^3, G^3] \leq [G,G,G^3]^9\gamma_5(G)^9. \tag{3.4.2}$$

Observe that $[G^3, G^3]^{3^{n-1}} \leq [G,G]^{3^{n+1}}\gamma_4(G)^{3^n} = 1$, by (3.4.1) and [16, Lemma 1.2.12(ii)], and similarly by (3.4.2), we have $[G^3, G^3, G^3]^{3^{n-2}} = 1$. □

Before stating the next theorem, we note that for a group G of exponent 3, $\exp(G \wedge G)$ divides $\exp(G)$ [27, Proposition 7(iii)], or alternatively by Corollary 2.5.

Theorem 3.5. *If G is a finite 3-group of class at most 5, then* $\exp(G \wedge G)$ *divides* $\exp(G)$.

Proof. We show that G satisfies the conditions of Lemma 3.3. Since groups of exponent 3 are of class at most 3 [17], we have $\gamma_4(G) \leq G^3$. Let $\exp(G) = 3^n$ and note that the case $n = 1$ is trivial, so we assume $\exp(G) = 3^n \geq 9$. Noting that $\gamma_4(G) \leq G^3$, (3.4.1) yields $[G^3, G^3] \leq (G^3)^3$, hence G^3 is powerful. Next to prove $\exp(G^3) = 3^{n-1}$, we show that $(x_1^3 \ldots x_k^3)^{3^{n-1}} = 1$ for all $x_i \in G$ and $k \geq 1$. We proceed by induction on k, the case $k = 1$ follows trivially. Let $k \geq 2$, induction hypothesis yields $(x_1^3 \ldots x_{k-1}^3)^{3^{n-1}} = 1$. Then setting $x = x_1^3 \ldots x_{k-1}^3$ and $y = x_k$ in Lemma 3.4 gives $(x_1^3 \ldots x_k^3)^{3^{n-1}} = 1$.

□

By Corollary 2.5, for $p \geq 5$ the p-groups of class 5 satisfy the conclusion of the above theorem. In particular, we have the following:

Corollary 3.6. *Let p be an odd prime. If G is a finite p-group of class at most 5, then exp$(G \wedge G)$ divides exp(G).*

4. Bounds depending on the nilpotency class and derived length

A group G is n-central if the exponent of $G\,/\,Z(G)$ divides n, and n-abelian if $(xy)^n = x^n y^n$ for all $x, y \in G$. In [26], Moravec showed that if a regular p-group G is p^n-central, then G is p^n-abelian. In [9], G. Ellis proved the existence of a covering pair for a pair of groups (G, N). Furthermore, he gave an isomorphism between $[N^*, G]$ and $N \wedge G$, where N^* is a covering pair of the pair (G, N). We use this isomorphism and the notions of n-central and n-abelian in the following lemma.

Lemma 4.1. *Let p be an odd prime and G be a finite p-group. If $N \trianglelefteq G$ of nilpotency class at most $p-2$, then* $\exp(N \wedge G) \mid \exp(N)$.

Proof. Consider a projective relative central extension $\delta : N^* \to G$ associated with the pair (G, N). Therefore $\delta(N^*) = N$ and G acts trivially on $ker(\delta)$. We know from [9] that $[N^*, G] \cong N \wedge G$. Since N is of class at most $p - 2$, N^* is of class at most $p-1$, therefore it is regular. Set $\exp(N) = t$, and since N^* is t-central, it is t-abelian [26]. The claim follows since $(n^{-1}n^g)^t = 1$ for all $n \in N^*$ and $g \in G$, as N^* is t-abelian and $n^t \in ker(\delta)$. □

In [10], Ellis proved that if the nilpotency class of G is c, then $\exp(M(G)) \mid (\exp(G))^{\lceil \frac{c}{2} \rceil}$. In [26], Moravec improves this bound by showing that $\exp(M(G)) \mid (\exp(G))^{2\lfloor \log_2 c \rfloor}$. Let p be an odd prime and G be a nilpotent group of class c, and $k = \lfloor \log_{p-1} c \rfloor$. Sambonet [36, Theorem 1.1] proved that $\exp(G \wedge G)$ divides $\exp(G)^{k+1}$. Observe that $k + 1 = \lceil \log_{p-1}(c+1) \rceil$ and $(p-1)^k \leq c \leq (p-1)^{k+1}$. The following result improves this bound when $c < (p+1)(p-1)^{k-1}$, and it offers an alternative proof for the general case.

Theorem 4.2. *If p is an odd prime and G is a p-group of class $c \geq p$, then* $\exp(G \wedge G) \mid (\exp(G))^n$, *for* $n = \lceil \log_{p-1}(\frac{c+1}{p+1}) \rceil + 1$.

Proof. For $c = p$, we have $\exp(G \wedge G) \mid \exp(G)$ by Corollary 2.5. Let $c > p$ and consider (1.0.4) with $N = G$ and $M = \gamma_m(G)$ where $m = \lceil \frac{c+1}{p-1} \rceil$. Then note that $\gamma_{p-1}(\gamma_m(G)) \leq \gamma_{c+1}(G) = 1$, hence Lemma 4.1 yields $\exp(\gamma_m(G) \wedge G) \mid \exp(\gamma_m(G))$. Observe that $\frac{c+1}{p+1} \leq (p-1)^{n-1}$, which implies that $\frac{c+1}{p-1} \leq (p-1)^{n-2}(p+1)$, in particular $m \leq (p-1)^{n-2}(p+1)$. Now applying induction hypothesis, we obtain $\exp[(G/\gamma_m(G)) \wedge (G/\gamma_m(G))] \mid (\exp(G/\gamma_m(G))^{n-1}$, and the result follows.

□

Lemma 4.3. *Let p be an odd prime and N be a normal subgroup of a finite p-group G. If the nilpotency class of N is c, then* $\exp(N \wedge G) \mid (\exp(N))^n$, *where* $n = \lceil \log_{p-1}(c+1) \rceil$.

Proof. We proceed by induction on c. If $c \leq p-2$, then by Lemma 4.1 $\exp(N \wedge G) \mid \exp(N)$. Let $c > p-2$ and take $M = \gamma_m(N)$ in (1.0.4), where $m = \lceil \frac{c+1}{p-1} \rceil$. Now the proof follows as in the above theorem, mutatis mutandis. □

Using the above Theorem and Proposition 3.2 yields an improvement on the exponent of the exterior square in Theorem 4.7 of [30].

Corollary 4.4. *Let p be an odd prime and G be a p-group of coclass r. Keeping the notations and assumptions of Theorem 4.7 of [30], we have* $\exp(G \wedge G) \mid p^{e+fn}$*, where* $n = 1 + \lceil log_{p-1}(\frac{m}{p+1}) \rceil$.

Proof. Consider (1.0.4) with $N = G$ and $M = \gamma_m(G)$. Now $\gamma_m(G)$ being powerful (cf. Theorem 1.2 of [37]) and class of $G/\gamma_m(G)$ being at most $m-1$, applying Proposition 3.2 and Theorem 4.2 yields the required bound. □

In [34] and [11], the authors give an isomorphism between the nonabelian tensor square of G and the subgroup $[G, G^\phi]$ of $\nu(G)$. We use this isomorphism in the proof of the next lemma. We also make use of the following identity which is a slight generalization of the identity (3.0.3) given by Ellis. Let $N, M \trianglelefteq G$. If $n \otimes [m, n, n] = 1$ and $n \otimes m, n \otimes [m, n]$ commute for all $n \in N, m \in M$, then

$$n^t \otimes m = (n \otimes [m, n])^{\binom{t}{2}}(n \otimes m)^t = (n^{\binom{t}{2}} \otimes [m, n])(n \otimes m)^t, \qquad (4.4.1)$$

for every $t \geq 2$.

Lemma 4.5. *Let G be a nilpotent group of class c and set $m = \lceil \frac{c+1}{3} \rceil$. If $\gamma_m(G)$ has odd exponent, then $exp(im(\gamma_m(G) \otimes G)) \mid exp(\gamma_m(G))$, where $im(\gamma_m(G) \otimes G)$ is the image of $\gamma_m(G) \otimes G$ in $G \otimes G$ under the natural map.*

Proof. Let $exp(\gamma_m(G)) = e$. Consider the isomorphism $\psi : G \otimes G \to [G, G^\phi]$ defined by $\psi(g \otimes h) = [g, h^\phi]$, where G^ϕ is an isomorphic copy of G. Recall that $[G, G^\phi]$ is a subgroup of $\nu(G)$. By Theorem A of [34], $\nu(G)$ is a group of nilpotency class at most $c+1$. Note that for $n \in \gamma_m(G), g \in G$, $\psi(n \otimes [g, n, n]) \in \gamma_{3m+1}(\nu(G))$. Since $3m+1 \geq c+2$, $\gamma_{3m+1}(\nu(G)) \subset \gamma_{c+2}(\nu(G)) = 1$. Thus $n \otimes [g, n, n] = 1$. Similarly $\psi([n, g] \otimes [n, [g, n]]) \subset \gamma_{3m+2}(\nu(G))$, giving $[(n \otimes g), (n \otimes [g, n])] = [n, g] \otimes [n, [g, n]] = 1$. Thus by (4.4.1), we obtain $n^e \otimes g = (n^{\binom{e}{2}} \otimes [g, n])(n \otimes g)^e$. Since e is odd, it follows that $e \mid \binom{e}{2}$, whence $(n \otimes g)^e = 1$. Moreover for $n_1, n_2 \in \gamma_m(G)$ and $g_1, g_2 \in G$, we have $\psi(\gamma_3(\langle n_1 \otimes g_1, n_2 \otimes g_2 \rangle)) \subset \gamma_{3m+3}(\nu(G)) = 1$. Hence $\langle n_1 \otimes g_1, n_2 \otimes g_2 \rangle$ is a group of class at most 2. Thus $((n_1 \otimes g_1)(n_2 \otimes g_2))^e = (n_1 \otimes g_1)^e (n_2 \otimes g_2)^e ([n_2, g_2] \otimes [n_1, g_1])^{\binom{e}{2}} = 1$, proving the result. □

Lemma 4.6. *Let G be a finite group with nilpotency class $c > 1$ and set $n = \lceil \log_3(\frac{c+1}{2}) \rceil$. If $\exp(G)$ is odd, then $\exp(G \wedge G) \mid (\exp(G))^n$. In particular, $\exp(M(G)) \mid (\exp(G))^n$.*

Proof. The proof is by induction on n. Note that $n \geq \log_3(\frac{c+1}{2})$ if and only if $c \leq (3^n \cdot 2) - 1$. When $n = 1$, the statement follows by Corollary 3.6. Now we proceed to prove it for n. Set $m = \lceil \frac{c+1}{3} \rceil$ and consider (1.0.3) with $N = G$ and $M = \gamma_m(G)$, which yields $\exp(G \wedge G) \mid \exp(im(\gamma_m(G) \wedge G)) \exp(\frac{G}{\gamma_m(G)} \wedge \frac{G}{\gamma_m(G)})$. By Lemma 4.5, we obtain that $\exp(im(\gamma_m(G) \wedge G)) \mid \exp(G)$. Now $\frac{G}{\gamma_m(G)}$ is a group of nilpotency class $m-1$. Since $c+1 \leq (3^n \cdot 2)$, $\frac{c+1}{3} \leq (3^{n-1} \cdot 2)$ giving $m \leq (3^{n-1} \cdot 2)$. Now applying induction hypothesis to the group $\frac{G}{\gamma_m(G)}$, we obtain $\exp(\frac{G}{\gamma_m(G)} \wedge \frac{G}{\gamma_m(G)}) \mid (\exp(G))^{n-1}$, which proves the lemma.

□

Corollary 4.7. *If G is a 3-group of nilpotency class $c \leq 17$, then $\exp(G \wedge G)$ divides $\exp(G)^2$.*

In a similar way, our technique which is based on Ellis's sequence (1.0.3) can be used to obtain Sambonet's [35] bound for $\exp(G \wedge G)$ for solvable groups.

Theorem 4.8 (Sambonet)**.** *If G is a finite solvable group of derived length d then*

i) If the order of G is odd, then $\exp(G \wedge G)$ divides $\exp(G)^d$.

ii) If the order of G is even, then $\exp(G \wedge G)$ divides $2^{d-1} \exp(G)^d$.

In fact, we have the following result of its own interest:

Lemma 4.9. *Suppose that N is a normal subgroup of a finite group G, which is solvable of derived length d.*

(i) If the order of N is odd, then $\exp(N \wedge G) \mid (\exp(N))^d$.

(ii) If the order of N is even, then $\exp(N \wedge G) \mid 2^d(\exp(N))^d$.

Proof. Take $M = N^{(d-1)}$ in (1.0.4), and use induction on d. The case $d = 1$ is Lemma 3.1.

□

The case $i)$ of Theorem 4.8 is simply the above Lemma with $N = G$, and the case $ii)$ follows by (1.0.4) with $N = G$ and $M = \gamma_2(G)$, since $\exp(\gamma_2(G) \wedge G)$ divides $2^{d-1} \exp(\gamma_2(G))^{d-1}$ by the above Lemma, and $\exp[(G/\gamma_2(G)) \wedge (G/\gamma_2(G))]$ divides $\exp(G/\gamma_2(G))$ since $G/\gamma_2(G)$ is abelian.

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