AN APPLICATION OF THE WEIGHTED LINEAR SIEVE FOR NUMBER FIELDS

A. PAN

ABSTRACT. A classical result of Y. Linnik from 1944 states that given a pair of positive integers a, q with $\gcd(a, q) = 1$, the least prime congruent to $a \bmod q$ is at most cq^L for effectively computable constants c and L. The Generalised Riemann Hypothesis implies that one can take $c = 1$ and $L = 2 + \epsilon$ for any real $\epsilon > 0$. In 2011, T. Xylouris established that one can take $c = 1, L = 5$ for $q \gg 1$. However, in 1969, H. E. Richert had showed if one considers the **least product of at most two primes** instead of the **least prime** in Linnik's result, one can take $c = 1$ and $L = 2.2$ for $q \gg 1$.

In our work, we consider a well known generalisation of this problem to number fields. The precise question is as follows. Given a number field $\mathbf{K}$, an integral ideal $\mathfrak{m}$ of $\mathbf{K}$, let $H_{\mathfrak{m}}(\mathbf{K})$ denote the Narrow ray class group of the number field $\mathbf{K}$ with respect to the ideal $\mathfrak{m}$. In 1983, A. Weiss proved that in any given class of $H_{\mathfrak{m}}(\mathbf{K})$, there exists **a prime ideal** of absolute norm bounded by $c_{\mathbf{K}}\mathfrak{N}_{\mathbf{K}/\mathbb{Q}}(\mathfrak{m})^{L_1}$ for effectively computable constants $c_{\mathbf{K}}$ (depending on $\mathbf{K}$) and L_1 absolute. In 2017, J. Thorner and A. Zaman proved that one can set $L_1 = 521$ in Weiss' result and gave a nearly explicit upper bound on $c_{\mathbf{K}}$ as well. That is, the upper bound is explicit upto some absolute constant. In 2025, J.- M. Deshouillers, S. Gun, O. Ramaré and J. Sivaraman showed that if one considers a **product of exactly three prime ideals** instead of **a single prime ideal**, one can set $L_1 = 9$ and give a fully explicit bound on $c_{\mathbf{K}}$.

In our work, we establish an analogue of Richert's result for quadratic number fields $\mathbf{K}$. Given a quadratic number field $\mathbf{K}$, we show that if one considers a **product of at most two prime ideals** instead of **a single prime ideal**, one can set $c_{\mathbf{K}} = 1$ and $L_1 = 3.8$ for $\mathfrak{N}_{\mathbf{K}/\mathbb{Q}}(\mathfrak{m})$ sufficiently large (depending on $\mathbf{K}$ alone). Unlike earlier results, our result is only for quadratic number fields, inexplicit and considers a product of at most 2 prime ideals instead of exactly 1 prime ideal or product of exactly 3 prime ideals, but it gives a lower exponent on $\mathfrak{N}_{\mathbf{K}/\mathbb{Q}}(\mathfrak{m})$. This is subject of joint work with Dr. J. Sivaraman.