A Study on Generalized Halmos Conjectures and Constrained Unitary Dilations

Pankaj Dey*

Abstract

The quadratic form associated with an operator on a Hilbert space restricted to the unit sphere of the Hilbert space is known as the numerical range. Halmos conjectured that the numerical range of a contraction T is the intersection of the numerical ranges of all unitary dilations of T. Durszt settled it in negative. Later in 2001, Choi and Li proved that the closure of the numerical range of a contraction T on a Hilbert space $\mathcal{H}$ is the intersection of the closure of the numerical ranges of all unitary dilations of T to $\mathcal{H} \oplus \mathcal{H}$. In 2014, Bercovici and Timotin refined this result by showing that for a contraction T with the same and finite defect indices, say, N, the closure of the numerical range of T is the intersection of the closure of the numerical ranges of all unitary N-dilations of T.

There are various generalizations of the numerical range in several different directions. One of them arises through the basic problem of error correction in quantum computation, which is known as the higher rank numerical range, introduced by Choi, Kribs and Życzkowski. For $1 \leq k \leq \infty$, the *rank-k numerical range* of an operator T is defined as the following set

$$\{\lambda \in \mathbb{C} : PTP = \lambda P, \text{ for some projection } P \text{ of rank } k\}.$$

Using the completely positive map, Arveson introduced a noncommutative analogue of the numerical range, known as the matricial range. For $1 \leq n < \infty$, the n^{th} *matricial range* of an operator T is defined as follows

$$\{\varphi(T) : \varphi : \operatorname{span}\{I, T, T^*\} \to M_n \text{ is unital completely positive}\}.$$

One more generalization of the numerical range that attracted a lot of attention is the C-numerical range, introduced by Westwick. Let $C \in M_n$ and T be an operator on an Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) \geq n$. The *C-numerical range* of T is defined as the following set

$$\{\operatorname{tr}[(C \oplus 0)U^*TU] : U \text{ is unitary on } \mathcal{H}\}.$$

This talk is based on the study carried over in my doctoral thesis. We first discuss a closure-free description of the higher rank numerical range of a normal operator using its spectral measure. Next, we move on to the detailed study of the problem of Halmos on various generalizations of the numerical range, viz, the higher rank numerical range, the C-numerical range and the matricial range of a contraction using its unitary dilations. These investigations give many interesting constrained unitary dilation results. An attempt is also made to an old problem on characterizing the matricial range of the Jordan block of size n for $n \geq 3$ and obtained some partial results in this direction.

*Reg No: IPHD15018, School of Mathematics, IISER Thiruvananthapuram.

References

[1] Dey, P., Mukherjee, M. *Higher rank numerical ranges of normal operators and unitary dilations.* **J. Math. Anal. Appl.** 524 (2023), no. 2, Article No. 127077, 16 pp. https://doi.org/10.1016/j.jmaa.2023.127077

[2] Dey, P., Mukherjee, M. *Generalized Halmos conjectures and constrained unitary dilations.* **Adv. Oper. Theory** 7 (2022), no. 4, Paper No. 56, 24 pp. https://doi.org/10.1007/s43036-022-00217-x

[3] Dey, P., Mukherjee, M. *Matricial range of the Jordan block of size three.* Manuscript under preparation.

[4] Dey, P., Mukherjee, M. *Matricial ranges and constrained unitary dilations.* Manuscript under preparation.